Lecture-10

One dimensional solution of Laplace’ Equation in spherical coordinate system

One dimensional solution of Laplace' Equation in spherical coordinate system

Next we consider the Laplace' Equation in spherical coordinates:

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}=0$$

In this system we consider that V is a function of r only.
Then the Laplace' equation reduces to

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial}\left(r^2\frac{\partial V}{\partial r}\right)=0 \qquad \text{--- (a)}$$

Again we exclude r = 0 from our solutions. Multiplying both Sides by r^2 we get

$$\frac{\partial}{\partial}\left(r^2\frac{\partial V}{\partial r}\right)=0 \qquad \text{--- (b)}$$

$$\therefore \quad r^2 \frac{\partial V}{\partial r} = A$$

$$\text{or} \quad \frac{\partial V}{\partial r} = \frac{A}{r^2} \qquad \text{--- (c)}$$

Integrating once again, we get

$$\text{or} \quad V = -\frac{A}{r} + B \qquad \text{--- (d)}$$

where A and B are arbitrary constants to be evaluated. This equation represents a family of equi – potential surfaces for r = constant.

Let us choose two such equipotential surfaces at r = a and R = b, b > a , such that at r = a, $V = V_a$ and at r = b, $V = V_b$

We immediately recognise that this is the example of concentric spheres or Spherical capacitor

$$V_a = -\frac{A}{a} + B \quad \text{--- (e)}$$

$$V_b = -\frac{A}{b} + B \quad \text{--- (f)}$$

Solving these two equations we get

$$A = \frac{V_a - V_b}{\left(\frac{1}{a} - \frac{1}{b}\right)}$$ and $$B = \frac{V_a \frac{1}{b} - V_b \frac{1}{a}}{\left(\frac{1}{b} - \frac{1}{a}\right)}$$

Substituting the values of A and B in equation (d), we get,

$$V = \frac{V_a - V_b}{\left(\frac{1}{a} - \frac{1}{b}\right)}\left(\frac{1}{r}\right) + \frac{V_a\left(\frac{1}{b}\right) - V_b\left(\frac{1}{a}\right)}{\left(\frac{1}{b} - \frac{1}{a}\right)}$$

$$or \qquad V = \frac{V_a - V_b}{\left(\frac{1}{a} - \frac{1}{b}\right)}\left(\frac{1}{r}\right) + \frac{V_b\left(\frac{1}{a}\right) - V_a\left(\frac{1}{b}\right)}{\left(\frac{1}{a} - \frac{1}{b}\right)} \qquad \text{--- (g)}$$

Let V_b = 0 Then equation (g) becomes

$$V = V_a \frac{\left(\frac{1}{r} - \frac{1}{b}\right)}{\left(\frac{1}{a} - \frac{1}{b}\right)} \quad \text{--- (h)}$$

Let us, next, follow our Five step procedure to determine the capacitance of the spherical capacitor

$$\vec{E} = -\nabla V = -\nabla \left(V_a \frac{\left(\frac{1}{r} - \frac{1}{b}\right)}{\left(\frac{1}{a} - \frac{1}{b}\right)} \right) = \frac{V_a}{\left(\frac{1}{a} - \frac{1}{b}\right)} \frac{1}{r^2} \quad \hat{r}$$

$$\vec{D} = \varepsilon \vec{E} == \frac{\varepsilon}{r^2} \frac{V_a}{\left(\frac{1}{a} - \frac{1}{b}\right)} \quad \hat{r}$$

$$\vec{D} = \vec{D}_S = D_S \hat{a}_S = D_N \hat{a}_N = \frac{\varepsilon}{r^2} \frac{V_a}{\left(\frac{1}{a} - \frac{1}{b}\right)} \qquad \hat{r}$$

We recognize that $D_S = D_N = \rho_S$ evaluated on any one of the capacitor surfaces. Choosing the surface with ρ = as our surface, we get,

$$D_N = \rho_S = \frac{\varepsilon}{a^2} \frac{V_a}{\left(\frac{1}{a} - \frac{1}{b}\right)}$$

Therefore the charge Q on the capacitor plate is

$$Q = \oint_S \rho_S dS = \oint_S \frac{\varepsilon}{a^2} \frac{V_a}{\left(\frac{1}{a} - \frac{1}{b}\right)} dS = \frac{\varepsilon}{a^2} \frac{V_a}{\left(\frac{1}{a} - \frac{1}{b}\right)} \oint_S dS$$

Therefore we get the expression for the spherical capacitor as

$$C=\frac{|Q|}{V_a}=\frac{\varepsilon}{a^2}\frac{V_a}{\left(\frac{1}{a}-\frac{1}{b}\right)}\frac{4\pi a^2}{V_a}=\frac{4\pi\varepsilon}{\left(\frac{1}{a}-\frac{1}{b}\right)}$$

$$\therefore \quad C = \frac{4\pi\varepsilon}{\left(\frac{1}{a}-\frac{1}{b}\right)}$$

For an isolated sphere , i.e., as b → we get

$$\therefore \quad C = 4\pi\varepsilon a$$

Finally let us consider V as a function of θ only . In this case
The Laplace's equation reduces to

$$\nabla^2 \mathbf{V} = \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial V}{\partial\theta} \right) = 0$$

We exclude $r = 0$ and $\theta = n\pi/2$, $n = 0, \pm 1, \pm 2, \pm 3, \ldots$

Multiplying both sides of the above equation by $r^2 \sin\theta$, we get

$$\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial V}{\partial\theta} \right) = 0$$

Integrating the equation with respect to θ we get,

$$\sin\theta \frac{\partial V}{\partial \theta} = A$$

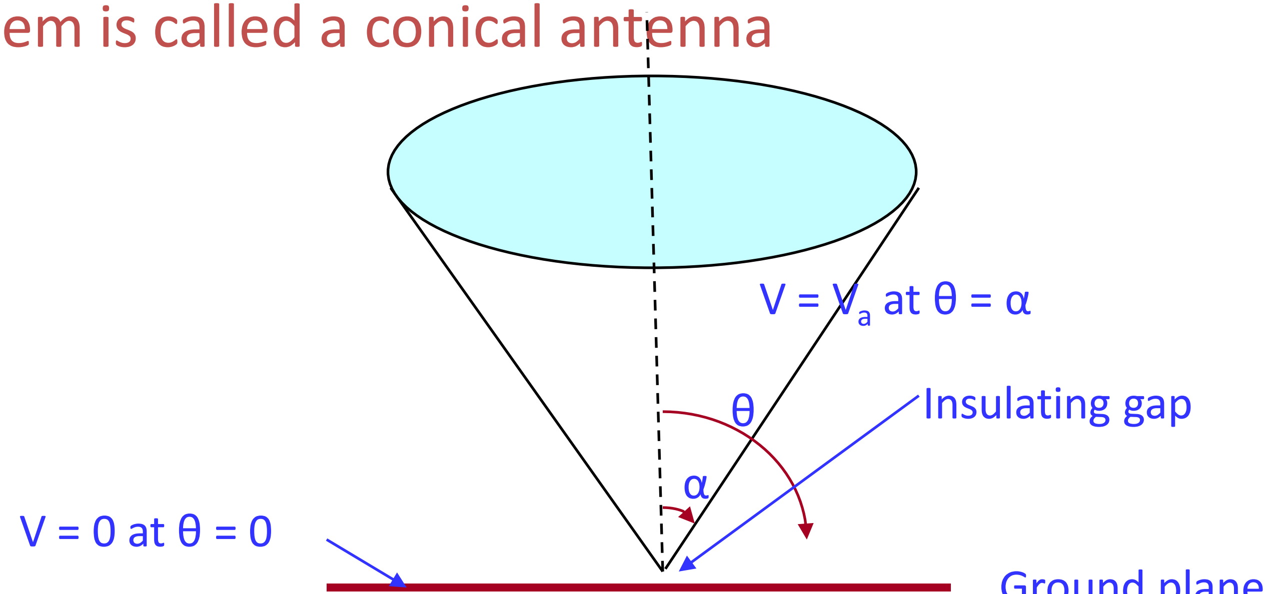
or $$\frac{\partial V}{\partial \theta} = \frac{A}{\sin\theta}$$ A is an arbitrary constant

Integrating once again, we get,

$$V = A\ln(\tan\theta/2) + B \qquad \text{--- (i)}$$

This equation represents a family of equipotential surfaces for constant θ. Let us consider two such equipotential surfaces at θ =π/2 and θ = α and let V = 0 at θ =π/2 and V = Va at θ =0.

The equipotential surfaces are cones as shown in figure below. Such a system is called a conical antenna



Applying these two boundary conditions to the equation (i), Solving for A and B and substituting these values in (i),we get,

$$V = V_a \frac{\ln(\tan\theta/2)}{\ln(\tan\alpha/2)}$$

We follow our usual procedure and determine the capacitance of the conical antenna

We have

$$V = V_a \frac{\ln(\tan\theta/2)}{\ln(\tan\alpha/2)}$$

We use E = - ∇ V to find the field strength, as

$$\vec{E} = -\nabla V = \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{a}_\theta = -\frac{V_a}{r\sin\theta\, ln(\tan\alpha/2)}\hat{a}_\theta$$

Next we determine **D** using **D** = ε_0 **E** as

$$\vec{D} = \varepsilon\vec{E} == -\frac{\varepsilon V_a}{r\sin\theta\, ln(\tan\alpha/2)}\hat{a}_\theta$$

D = D_S **a**$_S$ = D_N **a**$_N$, and $D_N = \rho_S$ and therefore, on the conical surface where θ = α, the charge density is,

$$\rho_S = -\frac{\varepsilon V_a}{r \sin\alpha \, ln(\tan\alpha/2)}$$

The total charge Q on the conical surface is therefore,

$$Q = \oint_S \rho_S ds = -\int_0^{\infty}\int_o^{2\pi} \frac{\varepsilon V_a}{r \sin\alpha \, ln(\tan\alpha/2)} r \sin\alpha \, d\phi dr$$

$$Q = -\frac{2\pi\varepsilon V}{ln(\tan\alpha/2)}_a \int_0^{\infty} dr$$

This equation leads to an infinite value of charge and capacitance.

Therefore we have to consider a cone of finite size.

Our expression for Q is approximate, since, theoretically, the potential surface θ= α extends from r = 0 to r = ∞. But our physical conical surface extends from r = 0 to say, $r = r_1$. The approximate capacitance is

$$C \doteq \frac{2\pi\varepsilon r_1}{ln(\mathbf{cot}\,\alpha/2)}_a$$